

Semiclassical Casimir Energies at Finite Temperature

Martin Schaden^{1,2} and Larry Spruch¹

¹New York University, Physics Department

4 Washington Place, New York, New York 10003

²Union College*, Physics Department

Science and Engineering Bldg., Schenectady, New York, 12308

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Abstract

We study the dependence on the temperature T of Casimir effects for a range of systems, and, in particular, for a pair of ideal parallel conductors, l_1 by l_2 , separated by a vacuum, a distance l_3 apart, with $l_1 \gg l_3$ and $l_2 \gg l_3$. We study the Helmholtz free energy A^T , combining Matsubara's formalism, in which the temperature T appears as a periodic Euclidean fourth dimension of circumference $l_T = \hbar c/k_B T$ with the semiclassical periodic orbital approximation of Gutzwiller. The latter was shown to be exact for parallel plates at $T = 0$. By inspecting the known results for the Casimir energy at $T = 0$ in two cases of a rectangular parallelepiped, ($l_1 \gg l_3$ and $l_2 \gg l_3$, and $l_1 \gg l_2$ and $l_1 \gg l_3$), one is led to guess at the expression for A^T of two ideal parallel conductors, without performing any calculation. The result is a new form for A^T , namely, $A^T = -(2\hbar c l_1 l_2 l_3 / \pi^2) \sum_{n_3=1}^{\infty} \sum_{n_T=-\infty}^{\infty} L^{-4}(n_3, n_T)$, where $L(n_3, n_T) = [(2n_3 l_3)^2 + (n_T l_T)^2]^{1/2}$ is the length of a classical periodic path on a two-dimensional cylinder section. This expression for the free energy is equivalent to others that have been obtained in the literature. At $T = 0$ the semiclassical approach provides a finite and systematic approximation scheme in terms of classical paths that is useful when the normal modes of the cavity cannot be determined either explicitly or implicitly. Slightly extending the domain of applicability of Gutzwiller's semiclassical periodic orbit approach, we here evaluate the free energy at $T > 0$ in terms of periodic classical paths in a four-dimensional cavity that is the tensor product of the original cavity and a circle. The validity of this approach is at present restricted to particular systems. We also discuss the origin of the classical (\hbar -independent) form of A^T for $T \sim \infty$.

Key Words: *Casimir energies, semiclassical theory, periodic orbits, finite temperature*

*Present address. Email address: schadenm@union.edu

1 INTRODUCTION

The best known of the Casimir effects is the force per unit area between two parallel ideal plates, separated by vacuum, a distance l_3 apart, at a temperature $T = 0$ [1]. Some fifty years later, that and other Casimir effects continue to intrigue both theoreticians and experimentalists. Indeed, interest has been growing. We list a few general references[2, 3, 4, 5]. Of particular interest to us here are recent experimental reports by Lamoreaux[6] and by Mohideen and colleagues[7]. Very recently the possible role of this effect in microelectromechanical systems has been examined[8]. To avoid alignment problems one studies the force between a sphere of radius R and a wall, where the point on the sphere closest to the wall is at a distance l from the wall, with $l \ll R$. (The analysis of the force in this case is intimately connected to the force between walls[2, 3, 9] and the experiment is simpler to perform. An experiment which might be even simpler to perform, because the force would be greater, is the force between a sphere and a segment of a spherical shell. We are here dealing with the focusing of virtual photons[10]. Focusing of virtual photons by a parabolic mirror has also been considered[11].) For the values of R and l studied, the accuracy was such that a number of corrections to the case of ideal conductors must be considered. In particular, finite temperature effects must be accounted for.

The corrections to the Casimir force for $T > 0$ were first studied in a seminal paper by Lifshitz[12] in 1956. The paper studied dielectric slabs, and the temperature effects were a matter of dispute for years. The subject was first clarified in 1978[13]. It was shown that Lifshitz's basic results for the temperature corrections were correct, but that Lifshitz had erred in taking the limit as the dielectric constant goes to infinity when going from a dielectric to an (ideal) conductor.

Our primary interest will be in rederiving the temperature effect for ideal walls but in a new form. We will utilize two sets of results, one by Matsubara[14] for accounting for temperature effects in a general setting, and the second a formulation of Casimir effects at $T = 0$ in terms of semiclassical extremal paths by the present authors[15]. The reformulation has the merit that the dominant paths are often rather obvious, with contributions that can be evaluated even when the proper frequency modes cannot be determined.

We take this opportunity to mention two significant papers[16] by Balian and Duplantier not referred to in[15]. The methods used in[16] were quite different from those used in our paper. We had not fully appreciated the two papers at the time[15] was published.

2 IDEAL PARALLEL PLATES: A HEURISTIC APPROACH

Matsubara showed[14] that in studying a system in three-dimensional space in equilibrium at temperature T , it can be convenient to consider the system in a four-dimensional Euclidean space whose fourth dimension at finite temperature $T > 0$ is compactified to a circle of circumference

$$l_T = \hbar c / (k_B T) . \tag{1}$$

Thus, consider two ideal parallel conductors at temperature T , with an area $\mathcal{A} = l_1 l_2$ separated a distance l_3 in vacuum with $l_3 \ll l_1$ and $l_3 \ll l_2$. The force between the plates in this case is proportional to l_1 and l_2 and the force per unit area does not depend on either dimension. At finite temperature, the essential geometry in this case is that of a two-dimensional cylindrical surface of length l_3 and circumference l_T . (See App. A)

In our study[15] of the plates at $T = 0$ the analysis involved the determination of the *difference*, $\Delta\rho(E)$, of the spectral densities in the presence and absence of the conductors. Gutzwiller[17] showed that this quantity is given in semi-classical approximation by the *periodic* paths of extremal *classical* action, in our case the periodic paths of extremal length. (Note that none of these paths has zero length and that a *stationary* periodic path must involve a minimum of two reflections.) In the case of parallel plates these are classical paths going back and forth between, and perpendicular to, the walls.

The analogous analysis at finite temperature would involve studying the semiclassical Green function with a periodic dependence on the Euclidean time. The initial and final space-time points are still the same, but the Euclidean geometry is that of a cylinder. We prefer to proceed differently.

For $T > 0$ (and fixed volume), it is natural to work with the Helmholtz free energy A^T rather than with \mathcal{E}^T , the (Casimir) energy itself, and there is a natural choice of the form of A^T . Thus, abandoning any pretense of rigor, we guess at the form of A^T – there remains the determination of an over-all numerical factor – and justify our choice of A^T afterwards.

The semi-classical path approach has advantages that go well beyond providing a simple physical picture of the origin of the Casimir effect at $T > 0$; as noted earlier, one can often determine the significant paths even when the natural frequency modes cannot be determined (neither explicitly nor, as in the case of the generalized argument theorem[18, 19], implicitly).

We have

$$A^T = A^T(\hbar, c, l_1, l_2, l_3, l_T). \quad (2)$$

At $T = 0$, we were concerned with the energy density. At $T > 0$, we will be concerned with the free energy density. In both cases we explicitly extract the volume $l_3 \mathcal{A}$ of the cavity. Further, on dimensional grounds, A^T is proportional to $\hbar c$. We can therefore write

$$A^T = \hbar c l_3 \mathcal{A} g(l_3, l_T), \quad (3)$$

where g is proportional to the inverse fourth power of a length. Now, in[15] it was shown that for a cavity with $l_3 \ll l_1, l_2$, \mathcal{E}_{Cas} is proportional to $\sum_{n_3} L^{-4}(n_3)$, where the length $L(n_3) = 2n_3 l_3$ and $n_3 = \pm 1, \pm 2, \dots$. Further, for a parallelepiped with $l_3 \ll l_1$ and $l_2 \ll l_1$, \mathcal{E}_{Cas} was proportional to $\sum_{n_2} \sum_{n_3} L^{-4}(n_2, n_3)$, where $L(n_2, n_3) = [(2n_2 l_2)^2 + (2n_3 l_3)^2]^{1/2}$ and the sum is over all pairs of integers (n_2, n_3) , positive, negative, and zero, other than $(0, 0)$. $L(n_3)$ is the length of a classical periodic path perpendicular to the walls and reflected $|n_3|$ times from each of them. Positive values of n_3 refer to paths which start to the right, while negative values of n_3 refer to paths which start to the left. $L(n_2, n_3)$ is the length of a classical periodic path with $|n_2|$ reflections off each of the walls with dimensions l_1 by l_3 and $|n_3|$ reflections off each of the walls with dimensions l_1 by l_2 . For the case with $l_3 \ll l_1$ and $l_3 \ll l_2$, $T > 0$, to which we henceforth largely restrict our attention, the knowledge that the space is enlarged by one

extra Euclidean dimension strongly suggests that A^T can be obtained by simply replacing $L(n_2, n_3)$ by

$$L(n_3, n_T) = [(2n_3l_3)^2 + (n_Tl_T)^2]^{1/2} \quad (4)$$

This length is that of a classical periodic extremal path on the surface of a cylinder of height l_3 and circumference l_T . The path starts in a given direction and reflects $|n_3|$ times from each of the ends of the cylinder section (the ideal walls) and circles the cylinder in a particular direction $|n_T|$ times. (The absence of a factor of 2 in the n_Tl_T term of Eq. (4) is due to the fact that we are considering extremal paths on a section of a *cylindrical* surface. See App. A) Negative values of n_3 refer to paths that start in the opposite direction, while negative values of n_T refer to paths that circle the cylinder in the opposite direction. The contribution from the $n_3 = 0$ path can be ignored since it gives an (extensive) contribution to the free energy that is proportional to the volume; it does not contribute to the (net) force on a plate. [See paragraph in Sec. 4 above the paragraph which contains Eq. (21).]

We thus guess that

$$A^T = K\hbar c\mathcal{A}l_3 \sum_{(n_3, n_T)} L^{-4}(n_3, n_T) , \quad (5)$$

where the sum extends over all pairs of integers (n_3, n_T) with $n_3 \neq 0$, the length $L(n_3, n_T)$ is given by Eq. (4), and K is a numerical coefficient. We determine the value of K by demanding that A^T reduce to

$$\mathcal{E}_{\text{Cas}} = -\frac{\pi^2\hbar c\mathcal{A}}{720l_3^3} \quad (6)$$

for $T \sim 0$. But $T \sim 0$ implies that $l_T \sim \infty$ which in turn implies that only $n_T = 0$ contributes to the sum in Eq. (5). For $T \sim 0$ we thus have

$$A^{T \sim 0} = K\hbar c\mathcal{A}l_3 2 \sum_{n_3=1}^{\infty} \frac{1}{(2n_3l_3)^4} = \frac{\pi^4 K\hbar c\mathcal{A}}{720l_3^3} . \quad (7)$$

(The factor of 2 reflects the fact that we are now summing over positive values only of n_3 .) Comparison with Eq. (6) gives $K = -1/\pi^2$ and we obtain

$$A^T = -\frac{2\hbar c\mathcal{A}l_3}{\pi^2} \sum_{n_3=1}^{\infty} \sum_{n_T=-\infty}^{\infty} L^{-4}(n_3, n_T) . \quad (8)$$

In App. B we show that Eq. (8) is indeed correct by comparing to a previous result. A proof from first principles, for a scalar field, is given in Sec. 5. The force per unit area at a temperature T is then given by

$$\frac{F^T}{\mathcal{A}} = \frac{1}{\mathcal{A}} \frac{\partial}{\partial l_3} A^T = -\frac{2\hbar c}{\pi^2} \sum_{n_3=1}^{\infty} \sum_{n_T=-\infty}^{\infty} \frac{3(2n_3l_3)^2 - (n_Tl_T)^2}{[(n_Tl_T)^2 + (2n_3l_3)^2]^3} \quad (9)$$

The somewhat complicated form of F^T as opposed to the relatively simple form of A^T shows that the free energy is not just a more fundamental object of study but also a more transparent one. Given A^T , F^T follows as in Eq. (9).

3 THE HIGH-T AND LOW-T EXPANSIONS

The free energy A^T is readily obtained from Eq. (8) for z large or small, where

$$z = 2l_3/l_T = 2l_3k_BT/(\hbar c) . \quad (10)$$

Isolating the $n_T = 0$ contribution, the free energy of Eq. (8) may be rewritten in the form

$$\frac{A^T(l_3)}{\mathcal{A}} = -\frac{\pi^2\hbar c}{720l_3^3} - \frac{(k_BT)^2}{\pi^2\hbar cl_3}\Delta(z) . \quad (11)$$

From Eq. (8) and the definition Eq. (4) we have that the dimensionless function $\Delta(z)$ is

$$\Delta(z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} [m^2z + n^2/z]^{-2} = \Delta(1/z) . \quad (12)$$

[To check Eq. (12), let $z \rightarrow 1/z$, and interchange m and n .] The reflection property[20], $\Delta(z) = \Delta(1/z)$, is quite powerful. It relates the high- and low-temperature regimes, or, equivalently, the large and small l_3 regimes. The expansion of $\Delta(z)$ for $z \gg 1$ is found by using the identity[21]

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} \equiv J(a)/a = \frac{1}{2a^2}(a\pi \coth(a\pi) - 1) = -\frac{1}{2a^2} + \frac{\pi}{2a} + \frac{\pi}{a} \sum_{n=1}^{\infty} e^{-2\pi na} . \quad (13)$$

Differentiating Eq. (13) with respect to a^2 and then replacing a by mz one obtains, after multiplication by z^2 and summation over m ,

$$\begin{aligned} \Delta(z) &= \sum_{m=1}^{\infty} \left[-\frac{1}{2m^4z^2} + \frac{\pi}{4m^3z} + \frac{\pi}{2m^3z} \sum_{n=1}^{\infty} (1 + 2\pi mnz) e^{-2\pi mnz} \right] \\ &= -\frac{\pi^4}{180z^2} + \frac{\pi}{4z} \zeta(3) + \frac{\pi}{2z} \sum_{n=1}^{\infty} e^{-2\pi nz} (1 + 2\pi nz) \sum_{n/m \text{ integer} > 0} m^{-3} , \end{aligned} \quad (14)$$

where $\zeta(3) = \sum_{m=1}^{\infty} m^{-3} = 1.202 \dots$. Using the expansion of Eq. (14) in Eq. (11) gives for the free energy per unit area, for $2l_3k_BT > \hbar c$, or $2l_3 > l_T$,

$$\frac{A^T}{\mathcal{A}} = -\frac{\zeta(3)k_BT}{8\pi l_3^2} - \frac{(k_BT)}{l_T l_3} \left(1 + \frac{l_T}{4\pi l_3}\right) e^{-4\pi l_3/l_T} + O\left(\frac{k_BT}{l_3^2} e^{-8\pi l_3/l_T}\right) . \quad (15)$$

Using the expansion of Eq. (14) with z replaced by $1/z$ in Eq. (11) gives, for $2l_3k_BT < \hbar c$, or $2l_3 < l_T$,

$$\frac{A^T}{\mathcal{A}} = -\frac{\pi^2\hbar c}{720l_3^3} - \frac{k_BT}{2\pi l_T^2} \zeta(3) + \frac{\pi^2 k_B T l_3}{45 l_T^3} - \frac{k_BT}{l_T l_3} \left(1 + \frac{l_3}{\pi l_T}\right) e^{-\pi l_T/l_3} + O\left(\frac{k_BT}{l_T l_3} e^{-2\pi l_T/l_3}\right) . \quad (16)$$

These low-T (Eq. (16)) and high-T (Eq. (15)) expansions of A^T/\mathcal{A} , first obtained by Sauer, by Mehra, and by Levin and Rytov[22], have also been reproduced in[13]. (The present derivation is somewhat simpler, to some extent because some of the earlier derivations were in a broader context, with dielectric walls or rather arbitrary surfaces. Note that Mehra[22] uses a convention for which an *attractive* force between the plates is taken to be positive.) See[3]

for a physical explanation of the contribution proportional to T^4 in the low-T expansion Eq. (16) of the free energy.

The expansions Eq. (16) and Eq. (15) are both exponentially good and the two limits are “dual” in the sense that since $\Delta(z) = \Delta(1/z)$, they *both* follow from Eq. (14). The relative error of these approximations is largest at the transition point, $2l_3k_BT = \hbar c$, or $z = 2l_3/l_T = 1$. (The l_3 dependence of A^T/\mathcal{A} is not well defined at the transition point, since, with n arbitrary, a factor of $(2l_3/l_T)^n \sim 1$ does not change the function appreciably near this point.) The exponential corrections all have the same sign and are no greater than $\sim 2.3\%$ at any value of l_3k_BT . Perhaps most significant experimentally is that the high-temperature limit is valid for l_3 rather large compared to $\hbar c/(2k_BT)$. This implies that the fall-off of the Casimir force at room temperature is proportional to l_3^{-3} rather than l_3^{-4} for separations l_3 rather large compared to 4 microns. Since the largest separation between the plates of early experiments was a few microns, it appears unlikely that a Casimir force proportional to $1/l^4$ could have been observed over the entire range. It should, however, be recalled that we have here considered an ideal metal rather than a real one. On the other hand, corrections due to finite conductivity *diminish* with increasing separation and are expected to be negligible compared to the temperature correction for separations exceeding a few microns[3, 23].

4 THE CLASSICAL FORM OF A^T FOR $T \rightarrow \infty$

The fact that for parallel plates A^T at sufficiently large T assumes a classical (\hbar -independent) form, as seen in Eq. (15), warrants some comment. We begin with the remark that A^T in the high-T limit is not generally \hbar -independent. Thus, for a thin smooth closed surface, one finds[16]

$$A^T = -\sigma k_B T \ln(k_B T l / \hbar c) , \quad (17)$$

where the dimensionless quantity σ is a function of *ratios* of the radii of curvature and l is an appropriately defined overall length scale. The logarithmic dependence of Eq. (17) on the overall length scale and $k_B T / \hbar c$ implies that the *change* of the free-energy does not depend on $\hbar c$ when all lengths are rescaled by the same amount, i.e., the “breathing mode” is always classical at high temperatures.

The fact that the free energy is a pure power law for parallel plates makes that case a bit special. If we *assume* that the high-T limit of the free energy of two parallel plates is classical, the power law dependence of A^T follows on dimensional grounds. (It follows from $A^T = A^T(k_B T, l_1, l_2, l_3)$ that $A^T = k_B T l_1 l_2 g(l_3)$, where $g(l_3)$ must be proportional to $1/l_3^2$.) One may argue for the existence of a classical high-T limit (and the absence of logarithmic corrections) from the contribution to the free energy of a single cavity mode of frequency ω_k . Writing

$$A^T = -\beta^{-1} \ln Z = \sum_k A_k^T = \sum_k \left[\frac{\hbar \omega_k}{2} + \beta^{-1} \ln(1 - e^{-\beta \hbar \omega_k}) \right] , \quad (18)$$

A_k^T for frequencies $\omega_k \ll k_B T / \hbar$ becomes

$$A_k^{T \sim \infty} = \frac{\hbar \omega_k}{2} + \beta^{-1} \ln[\beta \hbar \omega_k (1 - \frac{\beta \hbar \omega_k}{2} + O((\beta \hbar \omega_k)^2))] \sim k_B T \ln(\frac{\hbar \omega_k}{k_B T}) , \quad (19)$$

ignoring terms of order $\beta(\hbar\omega_k)^2$. Inserting the high-T expression Eq. (19) in Eq. (18) formally gives an asymptotic expansion of A^T . But since the frequencies of a cavity generally can be arbitrarily high and the neglected terms are not small for *all* frequencies at a fixed value of T , this expansion is valid only in the presence of a cutoff. However, for the *change* of the free energies of the cavity in two different configurations the mode sum is finite if the two configurations can be connected *adiabatically*, as, for instance, by moving an internal boundary arbitrarily slowly. An asymptotic expansion for large T of this difference does make sense. Under an adiabatic change of boundary conditions, the eigenmodes of the cavity change from ω_k to ω'_k , but the quantum numbers of a mode are conserved. Using Eq. (19) one obtains for the asymptotic difference of the free energies when $T \sim \infty$

$$\Delta A^T = A'^T - A^T \sim k_B T \sum_k \left[\ln\left(\frac{\omega'_k}{\omega_k}\right) \right]. \quad (20)$$

The sum in Eq. (20) converges if the difference of the free energies of the two configurations has a finite limit at $T = 0$, since a finite temperature does not induce new ultraviolet divergences. The expression Eq. (20) for the difference of the free energies at sufficiently high temperature (or better, for $k_B T l \gg \hbar c$, where l is a relevant length scale of the problem) is classical and notably does not depend on \hbar . In this limit the entropy S of the cavity is $S = -A^T/T$ and Eq. (20) states that the difference in the entropies of a cavity at high temperatures approaches a constant that does not depend on the quantum scale \hbar . (That the entropy of a bosonic gas in a cavity has a *finite* limit as $T \rightarrow \infty$ is compatible with dimensional reduction, that is, the high temperature limit is described by Euclidean quantum field theory in 3 dimensions with \hbar replaced by $i\beta$ [14].)

Although plausible, it is not possible to prove the existence of a finite high temperature limit using *only* classical arguments, because one can only compare the energy $k_B T$ with the frequencies of the cavity by introducing a quantity with the dimension of \hbar . If the asymptotic limit exists, and we have just argued that it does, it is independent of the value of \hbar one has chosen to make this comparison. On dimensional grounds, the classical contribution to the change in entropy from a single (independent) mode can then depend only on the frequency ratio. Further, since the contribution to the (change in) entropy of (independent) modes is additive, classically the (change) in entropy due to a single mode is proportional to $\ln(\omega'/\omega)$. Eq. (20) then follows as the classical change in the free energy of a cavity from the definition of k_B .

In studying the force between l_1 by l_2 plates at a separation l_3 one can evaluate the energy in the volume between the plates and in the infinite volume outside the plates and then determine the force by differentiation with respect to l_3 . It is a standard practice to avoid infinite volumes by considering a box of fixed volume¹ $l_1 \times l_2 \times L$ with an additional (movable) plate parallel to the l_1 by l_2 walls at a distance l_3 from one of the walls[24]. The total free energy of the box is the sum of the free energies of each of the two subvolumes: $A_{tot}^T(l_1, l_2, L; l_3) = A^T(l_1, l_2, l_3) + A^T(l_1, l_2, L - l_3)$. The force on the additional wall is $-\partial A_{tot}^T / \partial l_3$. Extensive contributions to the free energy of the subsystems (proportional to their volume) do not contribute to the dependence of A_{tot}^T on

¹There should be no confusion with our usage of L to also denote path lengths.

l_3 and therefore do not result in a (net) force on the middle wall. Note that generally only the *difference* of the free energy of the box compared to the free energy of the box in some “standard” configuration, (say with the additional wall at $l_3 = L/2$) i.e., $A_{diff}^T(l_1, l_2, L; l_3) = A_{tot}^T(l_1, l_2, L; l_3) - A_{tot}^T(l_1, l_2, L; L/2)$, is finite.

With these arguments, we can, purely classically, obtain the high temperature limit of the change in the free energy, ΔA^T , of a scalar field in a box of dimension $l_1 \times l_2 \times L$, as we move an additional wall from a position at a distance l_3 from one of the l_1 by l_2 sides to the middle of the box at $L/2$. Thus, from Eq. (20), we have

$$\Delta A_{class}^T = k_B T \mathcal{A} \int \frac{d^2 k_\perp}{(2\pi)^2} \frac{1}{2} \sum_{n=-\infty}^{\infty} \left[\ln \frac{\omega(\pi n/l_3, k_\perp)}{\omega(2\pi n/L, k_\perp)} + \ln \frac{\omega(\pi n/[L-l_3], k_\perp)}{\omega(2\pi n/L, k_\perp)} \right] \quad (21)$$

where $\omega(k_3, k_\perp) = c\sqrt{k_3^2 + k_\perp^2}$, with $k_3 = \pi n/\bar{l}_3$ where \bar{l}_3 is the appropriate distance from the wall. Using the integral representation

$$\ln[a/b] = \frac{1}{2} \int_0^\infty \frac{d\lambda}{\lambda} [e^{-\lambda b^2} - e^{-\lambda a^2}] , \quad (22)$$

for the logarithms in Eq. (21), the integration over k_\perp can readily be performed on using polar coordinates. In addition we use the reflection property,

$$\sum_{n=-\infty}^{\infty} e^{-n^2 \pi^2 x} = (\pi x)^{-1/2} \sum_{n=-\infty}^{\infty} e^{-n^2/x} , \quad (23)$$

to rewrite the resulting sums over n . [Eq. (23) and Eq. (12) both relate the values of a function at the argument x and at the “reflected” argument $1/x$. Eq. (23) is a particular case of Poisson’s summation formula.] We then obtain

$$\frac{\Delta A_{class}^T}{k_B T \mathcal{A}} = -\frac{1}{16\pi\sqrt{\pi}} \sum_{n=-\infty}^{\infty} \int_0^\infty \frac{d\lambda}{\lambda^{5/2}} \left[l_3 e^{-\frac{n^2 l_3^2}{\lambda}} + (L-l_3) e^{-\frac{n^2 (L-l_3)^2}{\lambda}} - L e^{-\frac{n^2 (L/2)^2}{\lambda}} \right] . \quad (24)$$

The $n = 0$ contribution vanishes and $\sum_{n=-\infty}^{\infty}$ reduces to $2 \sum_{n=1}^{\infty}$. The integrals with $n \neq 0$ converge term by term; with

$$\int_0^\infty \frac{d\lambda}{\lambda^{5/2}} e^{-\alpha/\lambda} = \frac{\pi^{1/2}}{2\alpha^{3/2}} , \quad (25)$$

we thereby arrive at

$$\Delta A_{class}^T = -k_B T \mathcal{A} \frac{1}{16\pi} \zeta(3) \left[\frac{1}{l_3^2} + \frac{1}{(L-l_3)^2} - \frac{8}{L^2} \right] , \quad (26)$$

which for $L \gg l_3$ is the leading term of Eq. (15) up to a factor of 2; the factor appears because we considered the scalar case here.[We strongly suspect that there may be a simpler *classical* derivation of Eq. (26) than the one given here.]

A related problem is that of the Casimir energy, at $T = 0$, of two spinless charged particles in free space. There is a classical contribution in that case too, but it is *not* the leading term and the question arises if the term can be evaluated using classical theory. The answer in this case is also positive[25].

An interesting point is that on dimensional grounds the absence of \hbar leads to the absence of c , though we are studying electromagnetic waves. There is no contradiction; the classical Maxwell equations have meaning in the non-relativistic ($c \rightarrow \infty$) limit[26]. Thus, for example, in the dipole approximation the classical Maxwell equations give a c -independent radiative transition rate.

5 THE SCALAR FIELD

In App. B we prove the validity of Eq. (8) by making a connection with the results of[13]. We here give an ab initio proof of that equation for a massless scalar field. The relation to the electromagnetic case is discussed in App. C (in a less broad context than that given, for example, in ref.[16]).

Thus, to substantiate some of the arguments of the main text, we consider a free massless scalar field ϕ that satisfies Dirichlet boundary conditions

$$\phi(x_i = 0) = \phi(x_i = l_i) = 0 , \quad (27)$$

at the boundaries of a box with spatial dimensions $l_1 \times l_2 \times \dots \times l_D$. The box serves as an infrared cutoff and guaranties a discrete spectrum. The dimension D of the box is arbitrary and can be used to dimensionally regularize the ultraviolet behavior of subsequent expressions by analytic continuation to non-integer dimensions. Additional boundary conditions may be imposed on the field inside the box. One could, for instance, demand that the field also vanish on a spherical shell located within the box, or, more appropriately for the Casimir force between plates, that it also vanish on a plane within the box.

The free energy A^T is related to the partition function Z by $A^T = -\beta^{-1} \ln Z$, where $\beta = 1/(k_B T)$. Formally, Z is

$$Z = \text{Tr} e^{-\beta H} , \quad (28)$$

where $H = \sum_{\mathbf{k}} E_{\mathbf{k}}(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + 1/2)$ is the free hamiltonian. $a_{\mathbf{k}}$ ($a_{\mathbf{k}}^\dagger$) destroys (creates) a quantum with quantum numbers \mathbf{k} in the (possibly rather complicated) cavity. Crucial for the following is that H [ignoring any (weak) self-coupling of the scalar field] commutes with the number operators $n_{\mathbf{k}} = a_{\mathbf{k}}^\dagger a_{\mathbf{k}}$ no matter how complicated the boundary conditions on the field may be. H can therefore be diagonalized in a basis that enumerates the number of quanta in each cavity mode and has the eigenvalues $\sum_{\mathbf{k}} E_{\mathbf{k}}(n_{\mathbf{k}} + 1/2)$ with each $n_{\mathbf{k}}$ a positive integer or 0. Evaluating the trace of Eq. (28) in this number basis, one has

$$\ln Z = \ln \left[\prod_{\mathbf{k}} \sum_{n_{\mathbf{k}}=0}^{\infty} e^{-\beta(n_{\mathbf{k}}+1/2)E_{\mathbf{k}}} \right] = - \sum_{\mathbf{k}} \ln [2 \sinh(\beta E_{\mathbf{k}}/2)] . \quad (29)$$

Instead of computing the free energy or partition function directly, it will be convenient to first consider the energy at temperature T ,

$$\mathcal{E}^T = \frac{\partial}{\partial \beta} (\beta A^T) = - \frac{\partial}{\partial \beta} \ln Z . \quad (30)$$

Differentiating Eq. (29), one finds

$$\mathcal{E}^T = \sum_{\mathbf{k}} E_{\mathbf{k}} \left[\frac{1}{2} + \sum_{m=1}^{\infty} e^{-m\beta E_{\mathbf{k}}} \right] . \quad (31)$$

For $E_{\mathbf{k}} > 0$ one can use Cauchy's theorem with a contour that runs just above the real axis and is closed by a large semicircle in the upper complex plane and write

$$E_{\mathbf{k}} \left[\frac{1}{2} + \sum_{m=1}^{\infty} e^{-m\beta E_{\mathbf{k}}} \right] = - \lim_{\epsilon \rightarrow 0^+} \frac{1}{\pi} \int_{-\infty+i\epsilon}^{\infty+i\epsilon} \frac{\lambda^2 d\lambda}{E_{\mathbf{k}}^2 + \lambda^2} \left[1/2 + \sum_{m=1}^{\infty} e^{im\beta\lambda} \right], \quad (32)$$

up to terms that do not depend on $E_{\mathbf{k}}$ nor on β ; in Eq. (32) we have dropped the contribution from the semicircular part of the contour. (For a sufficiently large radius Ω of the contour, this contribution is Ω/π up to terms that vanish for $\Omega \rightarrow \infty$.) The integral along the semicircular part of the contour thus does not depend on $E_{\mathbf{k}}$ nor T and can be absorbed in the overall normalization of the energy. Such constant contributions are irrelevant for the discussion of energy differences. With $\lambda = x + i\epsilon$ and x real, the term on the right hand side of Eq. (32), in square brackets, is [27]

$$Q \equiv \left[1/2 + \sum_{m=1}^{\infty} e^{im\beta(x+i\epsilon)} \right] = \frac{i}{2} \cot(\beta x/2) + \pi \sum_{n=-\infty}^{\infty} \delta(\beta x - 2\pi n) + O(\epsilon). \quad (33)$$

We can now evaluate the integral in Eq. (32). Apart from terms of order ϵ that vanish in the limit $\epsilon \rightarrow 0^+$, we can replace λ by x . Because $\cot(\beta x/2)$ is an odd function of x , the imaginary part of Q does not contribute to the integral. (This reflects the assumption that the cavity is ideal.) Formally, the energy of an ideal cavity at finite temperature is thus, from Eqs. (31), (32), and (33),

$$\begin{aligned} \mathcal{E}^T &= - \sum_{\mathbf{k}} \sum_{n=-\infty}^{\infty} \frac{(2\pi n)^2}{\beta^3} E^{-2}(\mathbf{k}, n) = - \frac{\partial}{\partial \beta} \sum_{\mathbf{k}} \sum_{n=-\infty}^{\infty} \ln E(\mathbf{k}, n) \\ &= - \frac{\partial}{\partial \beta} \ln \left[\prod_{\mathbf{k}} \prod_{n=-\infty}^{\infty} E^{-1}(\mathbf{k}, n) \right], \end{aligned} \quad (34)$$

where

$$E(\mathbf{k}, n) = [E_{\mathbf{k}}^2 + (2\pi n/\beta)^2]^{1/2}. \quad (35)$$

Since $\mathcal{E}^T = -(\partial/\partial\beta) \ln Z$, Eq. (34) determines the partition function up to an overall normalization, or equivalently, the entropy up to an overall constant (which can be determined by demanding that the entropy vanish at $T = 0$). Eq. (34) formally shows that the partition function is proportional to the square root of the inverse determinant of the differential operator $H_4^2 = h^2 - (\hbar c)^2 \frac{\partial^2}{\partial x_T^2}$, where x_T is the additional Euclidean coordinate in which the fields are periodic (with periodic length l_T) and h is the differential operator (single particle hamiltonian) with eigenvalues $E_{\mathbf{k}}$. The positive eigenvalues of H_4 , the $E(\mathbf{k}, n)$ given in Eq. (35), are the allowed energies of a massless scalar in the *four*-dimensional cavity constructed by augmenting every point of the original three-dimensional cavity by a circle of circumference l_T , that is, the 4-dimensional space is the tensor product of the cavity and a circle whose circumference is proportional to the inverse temperature. From Eq. (34) and Eq. (30) we see that the free energy of the cavity can be written formally as

$$A_{\text{cavity}}^T = \beta^{-1} \sum_{\mathbf{k}} \sum_{n=-\infty}^{\infty} \ln E(\mathbf{k}, n) = \frac{\hbar c}{l_T} \int_0^{\infty} dE' \rho_{\text{cavity}}^T(E') \ln(E'), \quad (36)$$

with

$$\rho_{\text{cavity}}^T(E') = \sum_{\mathbf{k}} \sum_{n=-\infty}^{\infty} \delta(E(\mathbf{k}, n) - E'). \quad (37)$$

Eq. (37) shows that $\rho_{\text{cavity}}^T(E')$ is the spectral density of the *four*-dimensional cavity constructed from the three-dimensional one in the above manner. Changes of $\rho_{\text{cavity}}^T(E')$ due to (adiabatic) deformation of the cavity, including a change of temperature, can then be approximated semi-classically using periodic classical rays of the *four*-dimensional cavity. This leads to a description of the free energy in terms of periodic paths in the *four*-dimensional cavity which we exploited when we guessed at the form of the free energy in Sec. 2.

To be somewhat more concrete, consider as an example a cubic cavity with sides of length \bar{l} . The momentum components for the mode with energy $E_{\mathbf{k}}$ are $k_i \hbar = n_i \pi \hbar / \bar{l}$, and the energies in the corresponding four-dimensional cavity at finite temperature are

$$E(\mathbf{k}, n) = \pi \hbar c \left[\sum_{i=1}^3 \left(\frac{n_i}{\bar{l}} \right)^2 + \left(\frac{n}{l_T/2} \right)^2 \right]; \quad (38)$$

as always, the length associated with the fourth (periodic) dimension, the analog of the lengths associated with the spatial dimensions, is $l_T/2$.

Note that Eq. (36) is formal in two respects: the integral over the energy diverges due to the ultraviolet behavior of the spectral density, and Eq. (36) specifies the free energy only up to (a generally similarly divergent) contribution proportional to T . Considerably more rigorous and physically relevant expressions can be obtained from Eq. (36) for *changes* in the free energy due to adiabatic deformations of the three-dimensional cavity. In physical terms the difference in the free energy is work that has to be done to change the boundary conditions for the field and should therefore be finite whenever it is physically feasible to do so. The ambiguity that Eq. (36) defines the entropy only up to a constant that does not depend on the temperature is removed by requiring that the entropy vanish at $T = 0$.

Let us finally remark that for $T \rightarrow 0$ and $E_{\mathbf{k}}$ fixed,

$$\lim_{\beta \rightarrow \infty} \sum_{n=-\infty}^{\infty} \beta^{-1} \ln[E_{\mathbf{k}}^2 + (2\pi n/\beta)^2]^{1/2} = E_{\mathbf{k}}/2, \quad (39)$$

up to an infinite constant that does not depend on $E_{\mathbf{k}}$ and therefore cancels in differences. [Eq. (39) follows upon subtracting the $E_{\mathbf{k}}$ -independent term $\ln[\mu^2 + (2\pi n/\beta)^2]^{1/2}$ from the summand in Eq. (39) and noting that the so regularized sum defines a convergent integral in the limit $\beta \rightarrow 0$ with the value $(E_{\mathbf{k}} - \mu)/2$. The dependence on the subtraction point μ can be absorbed in the overall energy normalization and cancels in energy differences.] At $T = 0$ Eq. (36) thus gives the usual expression for the Casimir energy. The classical periodic orbits of the four-dimensional cavity in this limit are just the periodic orbits of the three-dimensional cavity, since any path that winds about the fourth dimension becomes arbitrarily long as $T \rightarrow 0$. Note that whereas the spectral density ρ_{cavity}^T generally is proportional to the four-dimensional volume $l_T V_{\text{cavity}}$ the factor $1/l_T$ in Eq. (36) ensures that A^T generally will only be

proportional to the three-dimensional volume of the cavity (which we extracted explicitly in Sec. 2).

Although our starting point, Eq. (34) for the free energy, is the field theoretic one[14], the evaluation of this expression using the spectral density of a massless scalar in four spatial dimensions and, further, the semiclassical approximation to (changes in) this spectral density using classical periodic rays of the four-dimensional cavity is perhaps a bit unusual. This sketch of the relation between the free energy and the periodic classical rays of a corresponding four-dimensional cavity is the mathematical and perhaps even the physical basis for our guess of the free energy of two parallel plates in Sec. 2.

6 DISCUSSION

For ideal parallel plates, the semiclassical periodic orbital formalism gives a result for \mathcal{E}_{Cas} different in form but equivalent to that obtained by other methods. However, the approach provides a simpler physical picture, allows a simpler transition from \mathcal{E}_{Cas} to A^T , and gives a form simpler to evaluate numerically. It evaluates changes in the free energy due to changes in the temperature and in the cavity boundaries using the periodic classical paths in a four-dimensional cavity which is the tensor product of the original three-dimensional one and a circle. In a number of instances this amounts to replacing the contribution to the Casimir energy of each periodic classical path of length L_γ at $T = 0$ by the sum of contributions of periodic classical paths of length $L_\gamma^T(n_T) = [L_\gamma^2 + (n_T l_T)^2]^{1/2}$ that, in addition, wind about the extra fourth dimension n_T times.

Furthermore, preliminary research indicates that this considerable simplification may be present for a number of geometries, for scalar as well as for electromagnetic fields, and, at least for parallel plates, for dielectric walls. The simplification is a slight extension of the domain of applicability of Gutzwiller's semi-classical periodic orbit approach, allowing the method to be used for a number of systems at finite temperature. Generalizing the result for parallel plates, it appears that the periodic classical paths of various lengths can be interpreted as dual variables to the eigenfrequencies of a cavity: in rectangular cavities the two descriptions are related by Poisson's summation theorem.

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APPENDIX A: FINITE TEMPERATURE AND THE INTRODUCTION OF A FOURTH DIMENSION

As a vague indication of why the temperature can be treated by introducing a periodic fourth dimension, compare the standard operators $\exp(-iHt/\hbar)$ in quantum theory and $\exp(-\beta H)$ in statistical mechanics. One finds $\beta = it/\hbar$. The presence of i suggests periodicity, and we therefore drop the factor i and

replace t by a period τ , giving $\beta = \tau/\hbar$. For photons, we can replace τ by l_T/c , leading to $l_T = \hbar c/k_B T$. For parallel plates, F/A must be proportional to $l_1 l_2$, and the problem reduces to the determination of a function of l_3 and l_T . If one introduces periodic orbits between the plates, then the appropriate surface in l_3 and l_T is the surface of a cylinder of length l_3 and circumference l_T . Periodic classical paths involve lengths $2n_3 l_3$, for n_3 to and fro motions between the plates, and $n_T l_T$, for circling n_T times around the cylinder. Since the coordinate system is Cartesian, the length of the shortest orbit that circles the cylinder n_T times and goes back and forth between the plates n_3 times is just Eq. (4). For a proper derivation of the introduction of a fourth dimension, see [14] and Sec. 5.

APPENDIX B: AGREEMENT WITH A PREVIOUS RESULT

We now show that F^T/\mathcal{A} as given by Eq. (9) agrees with the force per unit area obtained previously [13]. Using Eqs. (3.8), (3.10), (3.11), (3.12), line 2 of (3.13), and (3.14) of [13], we find

$$\frac{F^T}{\mathcal{A}} = -\frac{1}{4\beta l_3^3} \sum_{n=-\infty}^{\infty} c(n\hbar c/(2k_B T l_3)) , \quad (\text{B1})$$

where

$$c(\alpha) = -\frac{\beta}{4\pi^2 \alpha l_3} \frac{d^2}{d\alpha^2} J(\alpha) \quad (\text{B2})$$

with

$$J(\alpha) = \int_0^\infty dy \frac{\sin(\alpha y)}{e^y - 1} . \quad (\text{B3})$$

We obtain a different form for $c(\alpha)$ from that obtained in [13] by expanding the denominator of Eq. (B3),

$$J(\alpha) = \text{Im} \sum_{m=1}^{\infty} \int_0^\infty dy e^{(i\alpha - m)y} = \sum_{m=1}^{\infty} \frac{\alpha}{m^2 + \alpha^2} , \quad (\text{B4})$$

in agreement with Eq. (13). Performing the differentiation leads to

$$c(\alpha) = \frac{\beta}{2\pi^2 l_3} \sum_{m=1}^{\infty} \frac{3m^2 - \alpha^2}{[m^2 + \alpha^2]^3} . \quad (\text{B5})$$

Insertion of Eq. (B5) with $\alpha = n\hbar c/(2k_B T l_3) = n l_T/(2l_3)$ in Eq. (B1) reproduces Eq. (9).

APPENDIX C: ELECTROMAGNETIC VERSUS SCALAR FIELDS

We are primarily interested in electromagnetic fields and we therefore comment on the relationship between the free energy A_{sc}^T of a scalar field and the free

energy of an electromagnetic field, denoted, as elsewhere, by A^T . The boundary conditions for scalar and electromagnetic fields are different, and further, for the latter case, both the condition $\mathbf{div} \mathbf{E} = 0$, which has no scalar field analog, and polarization must be accounted for; these three differences are not, of course, unrelated. Electrodynamics assumes different forms in spaces of a different number D of spatial dimensions and we will consider only $D = 3$. We will further limit our considerations to a rectangular box, with arbitrary dimensions l_1 by l_2 by l_3 .

As is well known, the condition $\mathbf{div} \mathbf{E} = 0$ reduces the number of independent components of \mathbf{E} to at most two and for a perfectly conducting box gives conditions on the quantum numbers n_1, n_2 and n_3 of the modes. There are two independent components if none of the n_i vanish, one if one of the n_i vanishes, and none if two of the n_i vanish. In the periodic orbital approach one considers classical ray's; in this case the condition $\mathbf{div} \mathbf{E} = 0$ implies certain phase changes of each of the two independent polarizations upon reflection.

We consider polarization effects for three cases, i) $l_1 \gg l_3$ and $l_2 \gg l_3$, ii) $l_1 \gg l_2$ and $l_1 \gg l_3$, and iii) arbitrary values of l_1, l_2 and l_3 . For case i), the orbitals pass back and forth between, and perpendicular to, the two l_1 by l_2 walls, the two states of polarization behave identically, and the effective electromagnetic response function, δg_{el} , is twice that of the relevant scalar response function δg_{sc} . For case ii), the situation is the same for orbitals which pass back and forth between the l_1 by l_3 or the l_2 by l_3 walls. For orbitals which reflect off four walls, the pair of l_1 by l_3 and the pair of l_2 by l_3 walls, the polarization vector $\hat{\epsilon}$ is unchanged on reflection for $\hat{\epsilon}$ perpendicular to the plane of scattering; $\hat{\epsilon}$ is changed on reflection for $\hat{\epsilon}$ in the plane of scattering, but returns to its original value at the end of the (planar) periodic orbit, and once again we have, effectively, $\delta g_{el} = 2\delta g_{sc}$. The situation is quite different for case iii), since periodic orbits that reflect off all six walls are *not* planar. [Periodic rays that reflect off all six walls cannot be planar, since no plane can cut all surfaces of the rectangular three dimensional volume.] To visualize one such non-planar orbit consider a classical ray that is close to and parallel to one of the spatial diagonals of the parallelepiped – not contained in any of the surfaces – and lies in one of the diagonal planes that contains the diagonal and two of the corners of the parallelepiped not connected by the diagonal. After reflection off all three walls at one of the corners, the ray, for symmetry reasons, is again parallel to the diagonal of the parallelepiped and in the same plane as before reflection at the corner. After reflecting off the other three walls at the diagonally opposite corner, the path of the ray closes to a periodic one. If the initial ray is chosen very close to the diagonal, the length of this periodic orbit at the very least is close to twice the length of the diagonal of the parallelepiped, i.e., $L(1, 1, 1) = [(2l_1)^2 + (2l_2)^2 + (2l_3)^2]^{1/2}$. The length of the periodic classical ray that reflects once off each pair of parallel walls in reality does not depend on how close to the diagonal the incident ray is chosen and thus *is* twice the length of the diagonal. The calculation would be much more complicated if that were not the case. More complicated non-planar periodic paths, that reflect a different number of times off each pair of parallel surfaces are obtained by changing the initial direction of the ray. For non-planar orbits there is a mixing of the two states of polarization and δg_{el} generally is not just $2\delta g_{sc}$; the contribution to δg_{el} of non-planar orbits is proportional to the trace of an orthogonal two-by-two polarization matrix. The existence of non-planar periodic orbits for case iii),

and the consequent non-diagonality of the polarization matrix, make less surprising the fact that \mathcal{E}_{Cas} in case iii) depends[28] not only on the 3-dimensional path lengths,

$$L(n_1, n_2, n_3) = [(2l_1n_1)^2 + (2l_2n_2)^2 + (2l_3n_3)^2]^{1/2}, \quad (\text{C1})$$

but also on the one-dimensional path lengths $L(n_i) = 2l_in_i$ for $i = 1, \dots, 3$. The presence of one-dimensional paths can also be associated with the restriction of the quantum numbers due to the boundary conditions. Thus for scalar fields satisfying Dirichlet boundary conditions in a rectangular cavity – the eigenfunctions are products of sines – none of the n_i quantum numbers vanish. Finally, we note that we stated in[15] that the semiclassical approach was exact if any of the ratios formed from l_1, l_2 and l_3 was arbitrarily large, as is the case for i) and ii) but not for iii). The restriction was necessary because we had assumed that $\delta g_{el} = 2\delta g_{sc}$.

We remark, parenthetically, that the four dimensional space characterized by l_1, l_2, l_3 and l_T , with three spatial dimensions and one periodic dimension (with a circumference proportional to the inverse temperature), is not equivalent to electrodynamics in four spatial dimensions; $\text{div } \mathbf{E} \neq 0$ for the latter space.

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